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Nonlinear Sigma Models on Statistical Manifolds

Equations of Motion.

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DOI: https://doi.org/10.65323/gf-lab.2025.001 • **Posted:** 2025-11-14 • **License:** CC BY 4.0 • **PDF:** Download

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Abstract

We derive the classical equations of motion for nonlinear sigma models where the base manifold is a statistical manifold. In this framework, the dynamics are governed by the Fisher-Rao metric, which emerges from principles of statistical distinguishability. We compute the explicit form of the Laplace-Beltrami operator for several fundamental probability distributions (Bernoulli, Gaussian, Categorical, etc.), finding that the underlying geometries are constant curvature spaces (spherical, flat, and hyperbolic).

1 Introduction

Nonlinear sigma models are a cornerstone of theoretical physics, describing maps between a base manifold and a target manifold with applications ranging from condensed matter to string theory [1, 2, 3]. In parallel, the field of information geometry provides the parameter space of any statistical model with a canonical Riemannian structure - the Fisher-Rao metric - which quantifies the distinguishability between probability distributions [4, 5, 6].

This paper synthesizes these two frameworks by constructing nonlinear sigma models where the base manifold is a statistical manifold endowed with its Fisher-Rao metric. In this construction, the geometry is not an ad hoc choice but an emergent property of the underlying statistical system. Our approach is complementary to recent work that places parameter space in the target, where demanding stability of inference can recover gravitational dynamics [7]. Here, we isolate the effects of the statistical geometry itself by having fields propagate on it, mapping to a simple, flat target space.

A central result is the derivation of the explicit equations of motion for fields propagating on these statistical bases. This requires the calculation of the associated Fisher geometries and actions, which we compute for five prototypical probability distributions. We find a landscape of constant curvature spaces: flat (Bernoulli, $R = 0$), positive (categorical triangle, $R = +1/2$), and multiple negative cases (Gaussian $R = -1$, shifted exponential $R = -2$, Gumbel $R = -12/\pi^2$).

The structure of these classical dynamics, set by the base curvature $R[g]$, has a direct quantum-level consequence. In two dimensions, a classically scale invariant field theory on a curved background develops a Weyl anomaly. For N free scalar fields on these Fisher bases, the stress-tensor trace obeys $\langle T^a{}_a \rangle = -\frac{N}{24\pi} R[g]$, directly tying statistical distinguishability to conformal field theory data. This is distinct from the well known Ricci-flow renormalization of a target metric [1]; in our model, the base geometry is a fixed background whose curvature sets the anomaly density.

The paper is organized as follows. Section 2 presents the general framework. Section 3 derives the explicit metrics and equations of motion for the five example geometries. Section 4 explores the quantum implications of this classical setup, focusing on the Weyl anomaly. Section 5 discusses physical interpretations, including a proof that constant kinetic coefficients in exponential families correspond to quadratic log partition functions. Section 6 contrasts our model with string theory and outlines future directions. We conclude in Section 7.

2 The Information Geometric Sigma Model Framework

We take a d-dimensional statistical manifold $(\mathcal{M}, g)$ with coordinates θ^a parameterizing densities $p(x;\theta)$. (We focus primarily on $d = 2$ unless stated otherwise.) The Fisher-Rao metric can be written as

$$g_{ab}(\theta) = \int dx\, p(x;\theta)\, \partial_a \ln p\, \partial_b \ln p = \mathbb{E}_\theta[\partial_a \ln p\, \partial_b \ln p], \tag{1}$$

under standard regularity (boundary terms vanish, including for moving-support models unless stated otherwise).

We define an NLSM for fields $\phi^\mu : \mathcal{M} \to \mathcal{T}$ mapping to a flat target with $G_{\mu\nu} = \delta_{\mu\nu}$. The action, equation of motion, and stress tensor are

$$S[\phi] = \frac{1}{2\alpha} \int d^d\theta \,\sqrt{g}\, g^{ab}\, \partial_a\phi^\mu \partial_b\phi^\nu \delta_{\mu\nu}, \tag{2}$$

$$\Delta_{\mathcal{M}}\phi^\mu \equiv \frac{1}{\sqrt{g}}\partial_a\big(\sqrt{g}\, g^{ab}\partial_b\phi^\mu\big) = 0, \tag{3}$$

$$T_{ab} = \alpha^{-1}\Big(\partial_a\phi^\mu\, \partial_b\phi^\nu\, \delta_{\mu\nu} - \tfrac{1}{2}\, g_{ab}\, g^{cd}\, \partial_c\phi^\mu\, \partial_d\phi^\nu\, \delta_{\mu\nu}\Big)\,. \tag{4}$$

Notation. Target-space indices are $\mu, \nu = 1, \ldots, N$ and are always contracted with $\delta_{\mu\nu}$. Base-manifold indices are $a, b, \ldots$. When a base coordinate coincides with a common "location" parameter (often denoted μ in statistics), we write it as m to avoid confusing it with the target index μ.

Terminology. We use "Fisher–Rao (information) metric" for the base metric g_{ab} and, when used adjectivally, abbreviate to "Fisher" (e.g., Fisher base, Fisher geometry).

3 Statistical Geometries: Explicit Examples

We compute Fisher–Rao metrics, scalar curvatures, and write the action densities and Laplace–Beltrami equations of motion explicitly for each case.

Distributions and parameter domains (for reference).

- **Bernoulli**: $X \in \{0,1\}$ with $\mathbb{P}(X{=}1) = p$, $\mathbb{P}(X{=}0) = 1-p$, $p \in (0,1)$.
- **Gaussian**: $p(x;m,\sigma) = \dfrac{1}{\sqrt{2\pi}\,\sigma}\exp\!\left[-\dfrac{(x-m)^2}{2\sigma^2}\right]$, $m \in \mathbb{R}$, $\sigma > 0$.
- **Categorical (N=3)**: $(p_1,p_2,p_3) = (x, y, 1-x-y)$ with $x>0$, $y>0$, $x+y<1$.
- **Shifted exponential**: $p(x;m,\lambda) = \lambda e^{-\lambda(x-m)}\,\mathbf{1}_{x\ge m}$, $m \in \mathbb{R}$, $\lambda > 0$.
- **Gumbel**: $p(x;m,\beta) = \frac{1}{\beta}\exp\!\left[-\frac{x-m}{\beta} - e^{-(x-m)/\beta}\right]$, $m \in \mathbb{R}$, $\beta > 0$.

3.1 Bernoulli (1D, Flat)

For $p \in (0,1)$:

$$g_{pp} = \frac{1}{p(1-p)}, \qquad g^{pp} = p(1-p), \qquad \sqrt{g} = \big[p(1-p)\big]^{-1/2}, \qquad R = 0. \tag{5}$$

Action density:

$$\mathcal{L} = \frac{1}{2\alpha}\sqrt{g}\,g^{pp}\,\partial_p\phi^\mu\partial_p\phi^\nu\delta_{\mu\nu} = \frac{1}{2\alpha}\,\big[p(1-p)\big]^{1/2}\,\partial_p\phi^\mu\partial_p\phi^\nu\delta_{\mu\nu}. \tag{6}$$

EOM:

$$\Delta\phi^\mu = \frac{1}{\sqrt{g}}\partial_p\Big(\sqrt{g}\,g^{pp}\partial_p\phi^\mu\Big) = (p(1-p))^{1/2}\,\partial_p\Big((p(1-p))^{1/2}\partial_p\phi^\mu\Big) = 0, \tag{7}$$

equivalently,

$$p(1-p)\,\partial_p^2\phi^\mu + \tfrac{1}{2}(1-2p)\,\partial_p\phi^\mu = 0. \tag{8}$$

Action (specialized to Bernoulli):

$$S[\phi] \;=\; \frac{1}{2\alpha}\int_0^1 dp\;\sqrt{g}\,g^{pp}\,\partial_p\phi^\mu\,\partial_p\phi^\nu\,\delta_{\mu\nu} \;=\; \frac{1}{2\alpha}\int_0^1 dp\;\sqrt{p(1-p)}\,\Big(\frac{d\phi^\mu}{dp}\Big)^2\delta_{\mu\nu}. \tag{9}$$

Interpretation: "dynamical mass" along the Bernoulli line. The kinetic weight $\sqrt{g}\,g^{pp} = \sqrt{p(1-p)}$ plays the role of a position dependent inertia (or "mass density") for the field profile $\phi^\mu(p)$. Writing the Lagrangian as $\mathcal{L} = \frac{1}{2\alpha}\,m(p)\,(\partial_p\phi)^2$ with $m(p) = \sqrt{p(1-p)}$, the associated flux $J^\mu = \sqrt{p(1-p)}\,\partial_p\phi^\mu$ is conserved ($\partial_p J^\mu = 0$). Because the base is one–dimensional, this variable "mass" can be removed by a coordinate redefinition: define $u(p)$ via

$$\frac{du}{dp} = \frac{1}{\sqrt{p(1-p)}}, \qquad u(p) = 2\arcsin\sqrt{p} \in (0,\pi).$$

Then $S = \frac{1}{2\alpha}\int du\,(\partial_u\phi^\mu)^2\delta_{\mu\nu}$ and $\Delta\phi = \partial_u^2\phi$. In $d \ge 2$ (e.g. Gaussian, categorical), curvature/anisotropy obstructs such a global removal, so the position dependence is physical.

3.2 Gaussian (2D, Hyperbolic)

For $m \in \mathbb{R}$, $\sigma > 0$:

$$ds^2 = \frac{1}{\sigma^2} dm^2 + \frac{2}{\sigma^2} d\sigma^2, \qquad \sqrt{g} = \frac{\sqrt{2}}{\sigma^2}, \qquad g^{mm} = \sigma^2, \quad g^{\sigma\sigma} = \frac{\sigma^2}{2}. \tag{10}$$

Action density:

$$\mathcal{L} = \frac{1}{2\alpha} \sqrt{g} \Big(g^{mm}\, \partial_m \phi^\mu\, \partial_m \phi^\nu + g^{\sigma\sigma}\, \partial_\sigma \phi^\mu\, \partial_\sigma \phi^\nu \Big) \delta_{\mu\nu} = \frac{\sqrt{2}}{2\alpha} \Big(\partial_m \phi^\mu\, \partial_m \phi^\nu + \tfrac{1}{2}\, \partial_\sigma \phi^\mu\, \partial_\sigma \phi^\nu \Big) \delta_{\mu\nu}. \tag{11}$$

EOM:

$$\Delta \phi^\mu = \frac{1}{\sqrt{g}} \Big[\partial_m (\sqrt{g}\, g^{mm} \partial_m \phi^\mu) + \partial_\sigma (\sqrt{g}\, g^{\sigma\sigma} \partial_\sigma \phi^\mu) \Big] = \sigma^2 \Big(\partial_m^2 + \tfrac{1}{2}\, \partial_\sigma^2 \Big)\, \phi^\mu = 0. \tag{12}$$

and $R = -1$.

Geometric check: $(x = m/\sqrt{2},\ y = \sigma) \Rightarrow ds^2 = 2(dx^2 + dy^2)/y^2$.

3.3 Categorical, $N = 3$ (2D, Positive Curvature)

General categorical Fisher metric. For N outcomes with probabilities $p_i(\theta)$ obeying $\sum_i p_i = 1$, the Fisher metric in coordinates θ^a is

$$g_{ab}(\theta) = \sum_{i=1}^{N} \frac{1}{p_i(\theta)}\, \partial_a p_i(\theta)\, \partial_b p_i(\theta),$$

restricted to the open simplex $p_i > 0$.

With $(p_1, p_2, p_3) = (x, y, 1 - x - y)$:

$$g = \begin{pmatrix} \frac{1}{x} + \frac{1}{1-x-y} & \frac{1}{1-x-y} \\ \frac{1}{1-x-y} & \frac{1}{y} + \frac{1}{1-x-y} \end{pmatrix}, \qquad g^{-1} = \begin{pmatrix} x(1-x) & -xy \\ -xy & y(1-y) \end{pmatrix}, \tag{13}$$

$$\sqrt{g} = (xyz)^{-1/2}, \quad z = 1 - x - y, \qquad R = \tfrac{1}{2}. \tag{14}$$

Action density:

$$\mathcal{L} = \frac{1}{2\alpha} (xyz)^{-1/2} \Big[x(1-x)\, \partial_x \phi^\mu\, \partial_x \phi^\nu + y(1-y)\, \partial_y \phi^\mu\, \partial_y \phi^\nu - 2xy\, \partial_x \phi^\mu\, \partial_y \phi^\nu \Big] \delta_{\mu\nu}. \tag{15}$$

EOM (divergence form):

$$\Delta \phi^\mu = (xyz)^{1/2} \Big\{ \partial_x \Big[(xyz)^{-1/2} (x(1-x)\partial_x - xy\, \partial_y) \phi^\mu \Big] + \partial_y \Big[(xyz)^{-1/2} (- xy\, \partial_x + y(1-y)\partial_y) \phi^\mu \Big] \Big\} = 0. \tag{16}$$

Geometric check: $u_i = 2\sqrt{p_i}$ embeds the simplex as an octant of S^2 (radius 2), hence $R = 1/2$.

Remark (robustness under splitting a category). If one embeds a 2D submanifold of a 4–outcome model by splitting the leftover probability as $p_3 = c(1 - x - y)$, $p_4 = (1 - c)(1 - x - y)$

with fixed $c \in (0,1)$, the induced 2×2 Fisher metric in (x, y) is

$$g_{ab} = \begin{pmatrix} \frac{1}{x} + \frac{1}{1-x-y} & \frac{1}{1-x-y} \\ \frac{1}{1-x-y} & \frac{1}{y} + \frac{1}{1-x-y} \end{pmatrix},$$

independent of c. Thus the $N{=}4$ split surface is isometric to the $N{=}3$ simplex case.

3.4 Shifted Exponential (2D, Negative Curvature)

For $p(x; m, \lambda) = \lambda e^{-\lambda(x-m)} \mathbf{1}_{x \geq m}$, $\lambda > 0$:

$$g = \mathrm{diag}(\lambda^2, \lambda^{-2}), \qquad g^{mm} = \lambda^{-2}, \quad g^{\lambda\lambda} = \lambda^2, \qquad \sqrt{g} = 1, \qquad R = -2. \tag{17}$$

Action density:

$$\mathcal{L} = \frac{1}{2\alpha}\Big(\lambda^{-2}\, \partial_m \phi^\mu\, \partial_m \phi^\nu + \lambda^2\, \partial_\lambda \phi^\mu\, \partial_\lambda \phi^\nu\Big)\delta_{\mu\nu}. \tag{18}$$

EOM:

$$\Delta\phi^\mu = \partial_m(\lambda^{-2}\partial_m\phi^\mu) + \partial_\lambda(\lambda^2\partial_\lambda\phi^\mu) = 0, \qquad \text{equivalently} \qquad \lambda^{-2}\partial_m^2\phi^\mu + \lambda^2\partial_\lambda^2\phi^\mu + 2\lambda\, \partial_\lambda\phi^\mu = 0. \tag{19}$$

Moving support: the boundary at $x \geq m$ can be regularized by replacing the step with a smooth cutoff and taking $\varepsilon \to 0$; boundary terms vanish in this limit.

3.5 Gumbel (2D, Negative Curvature)

For location $m \in \mathbb{R}$ and scale $\beta > 0$, with Euler's constant γ_E:

$$g = \frac{1}{\beta^2} M, \quad M = \begin{pmatrix} 1 & \gamma_E - 1 \\ \gamma_E - 1 & (\gamma_E - 1)^2 + \pi^2/6 \end{pmatrix}, \qquad g^{ab} = \beta^2 M^{-1\,ab}, \qquad \sqrt{g} = \frac{\sqrt{\det M}}{\beta^2}, \qquad R = -\frac{12}{\pi^2}. \tag{20}$$

Action density:

$$\mathcal{L} = \frac{1}{2\alpha}\sqrt{g}\, g^{ab}\, \partial_a\phi^\mu\, \partial_b\phi^\nu\, \delta_{\mu\nu} = \frac{\sqrt{\det M}}{2\alpha}\, M^{-1\,ab}\, \partial_a\phi^\mu\, \partial_b\phi^\nu\, \delta_{\mu\nu}. \tag{21}$$

which is independent of β. *EOM:*

$$\Delta\phi^\mu = \frac{1}{\sqrt{g}}\partial_a\big(\sqrt{g}\, g^{ab}\partial_b\phi^\mu\big) = \beta^2 M^{-1\,ab}\, \partial_a\partial_b\phi^\mu = 0, \tag{22}$$

since $\sqrt{g}\, g^{ab}$ is constant.

3.6 Summary

We present a summary of previous results in Table 1.

Table 1: Fisher–Rao (base) geometries for representative distributions. All curvatures are constant.

Distribution	Parameters	Dim.	Geometry	R	EOM
Bernoulli	$p \in (0,1)$	1	Flat line	0	Eq. (7)
Gaussian	(m, σ)	2	Hyperbolic plane ($\mathbb{H}^2$)	-1	Eq. (12)
Categorical	(x, y)	2	Spherical simplex (octant of S^2)	$+1/2$	Eq. (16)
Shifted exp.	(m, λ)	2	Hyperbolic plane (scaled)	-2	Eq. (19)
Gumbel	(m, β)	2	Hyperbolic plane (scaled)	$-12/\pi^2$	Eq. (22)

4 Quantum Properties and Conformal Anomaly

4.1 Weyl Anomaly on Constant Curvature Bases

For N free target scalars on a curved two-dimensional base $(\mathcal{M}, g_{ab})$, the classical action, Equation (2), is scale invariant. Quantum mechanically, the conformal symmetry is anomalous. The trace acquires [8, 9, 10, 11]

$$\langle T^a{}_a \rangle = -\frac{N}{24\pi} R[g], \tag{23}$$

with central charge $c = N$ and anomaly density set by the geometry. In two dimensions, Gauss Bonnet gives

$$\int d^2\theta \sqrt{g}\, R = 4\pi\, \chi(\mathcal{M}), \tag{24}$$

so on compact bases (or with appropriate boundary conditions) the integrated anomaly is topological.

4.2 Covariant Background-Field Expansion and Constant Curvature

Following the background-field method (see, e.g., [12]), we expand the base metric around a point θ_0 using Riemann normal coordinates $\theta^a = \theta_0^a + \xi^a$:

$$g_{ab}(\theta_0 + \xi) = g_{ab}(\theta_0) + \frac{1}{3} R_{acbd}(\theta_0)\, \xi^c \xi^d + \frac{1}{6} (\nabla_c R_{adbe})(\theta_0)\, \xi^c \xi^d \xi^e + \mathcal{O}(\xi^4). \tag{25}$$

For constant sectional curvature, $R_{abcd} = \frac{R}{2(d-1)}(g_{ac} g_{bd} - g_{ad} g_{bc})$ and $\nabla R_{abcd} = 0$. Hence the leading nontrivial corrections are quadratic in ξ; higher orders begin at $\mathcal{O}(\xi^4)$ and are curvature–squared. This makes perturbation theory particularly simple on the constant-curvature Fisher bases in Section 3 (Gaussian, categorical, shifted–exponential, Gumbel), where R is uniform.

Split invariance (background independence). The decomposition $\theta = \theta_0 + \xi$ is arbitrary. An infinitesimal shift of the background, $\delta_v \theta_0^a = v^a$, can be compensated by a nonlinear transformation of ξ^a so that the total coordinate θ^a is unchanged:

$$\delta_v \xi^a \;=\; -v^a \;-\; \Gamma^a{}_{bc}(\theta_0)\, \xi^b v^c \;-\; \frac{1}{3} R^a{}_{bcd}(\theta_0)\, \xi^b \xi^d v^c \;+\; \mathcal{O}(\xi^3). \tag{26}$$

Because $\delta_v \theta^a = \delta_v \theta_0^a + \delta_v \xi^a = 0$, the classical action, Equation (2), is invariant under the split transformation order by order in ξ. Practically, Equation (26) ensures the background-field expansion is covariant and that counterterms organize into curvature tensors.

Link to the anomaly. The covariant organization in Equation (25) is precisely what underlies the heat-kernel coefficient $\frac{1}{6}R$ and the Weyl anomaly density, Equation (23). On constant-curvature Fisher bases, the absence of ∇R terms and the uniformity of R make the anomaly density spatially constant, matching the simplified structure of the expansion.

4.3 Implications

For constant curvature Fisher bases, the anomaly density is uniform. For non-compact cases (e.g. $\mathbb{H}^2$) the total anomaly requires IR regularization; locally, Equation (23) applies pointwise. On compact bases,

$$\int d^2\theta \sqrt{g}\, \langle T^a{}_a \rangle = -\frac{N}{24\pi} \int \sqrt{g}\, R = -\frac{N}{6}\, \chi(\mathcal{M}),$$

so the integrated anomaly is topological.

5 Physical Interpretation of the Dynamics

5.1 Exponential Families and Constant Kinetic Coefficients

For $p(x;\theta) = \exp[C(x) + \theta^a F_a(x) - \psi(\theta)]$, in natural coordinates

$$g_{ab}(\theta) = \partial_a \partial_b \psi(\theta). \tag{27}$$

Proposition 1 (Constant kinetic coefficient $\Leftrightarrow$ quadratic log-partition)**.** *For an exponential family, the following are equivalent:*

1. *$g_{ab}(\theta)$ are constant;*
2. *$\psi(\theta) = \frac{1}{2}A_{ab}\theta^a\theta^b + B_a\theta^a + C$ with constant A_{ab}, B_a, C.*

Proof. If ψ is quadratic then $\partial_a\partial_b\psi = A_{ab}$ is constant. Conversely, if all second derivatives are constant, integrating twice yields a quadratic plus affine ψ. □ □

Corollary (flat base and anomaly). If g_{ab} is constant in natural coordinates, the Levi–Civita connection has vanishing Christoffels and the base is (locally) flat ($R = 0$). Hence the local Weyl anomaly density vanishes by Equation (23).
Example. For the normal distribution with known variance σ_0^2 (one parameter, m unknown), the natural parameter is $\theta = m/\sigma_0^2$ and $\psi(\theta) = \frac{1}{2}\sigma_0^2\theta^2$, so $g_{\theta\theta} = \sigma_0^2$ is constant.
Remark. Quadratic ψ is a special case of Hessian (dually flat) geometry with constant Hessian; generically ψ is strictly convex but non-quadratic, leading to position-dependent kinetic weights and (in $d \geq 2$) nonzero curvature.

5.2 Anisotropic Scaling

When metric components scale differently, anisotropic theories arise. The Gaussian case becomes isotropic after rescaling; others can exhibit Lifshitz-like behavior.
Remark. In d=1 (Bernoulli) the position–dependent kinetic weight can be flattened by a reparametrization $p \to u(p)$ that renders the action free; for $d \geq 2$, curvature obstructs a global removal, so anisotropy is genuinely dynamical.

6 Relation to String Theory and Inference Driven Models

Our 2D sigma model invites comparison with string theory, but differs in signature (Riemannian Fisher metric vs. Lorentzian worldsheet), constraints (no Virasoro constraints here), and roles of geometry (base is fixed; target is flat). Work placing parameter space in the target - and recovering gravity from inference stability - is complementary [7]. Outlook: gauging base isometries (e.g. translations in m) and introducing target space gauge fields are natural extensions; a full renormalization analysis is left for future work.

Remark (Weyl anomaly vs. string criticality). In our background metric setup, the Weyl anomaly for N free scalars on the Fisher base is given by Equation (23). This does not fix N: the theory is consistent for any N, with a local anomaly density set by the base curvature. A "critical N" arises only if one integrates over metrics and gauges worldsheet diffeomorphisms and Weyl symmetry (string theory). In conformal gauge the Faddeev Popov bc ghosts contribute $c_{\text{ghosts}} = -26$, and quantum Weyl invariance imposes $c_{\text{matter}} + c_{\text{ghosts}} = 0$, giving $N = 26$ for N free bosons [8, 9, 10, 11]. By contrast, with a fixed Fisher base there are no ghosts and no criticality condition; one may still have a vanishing integrated anomaly on compact bases with $\chi(\mathcal{M}) = 0$ by Gauss Bonnet, since

$$\int d^2\theta \sqrt{g}\, \langle T^a{}_a \rangle = -\frac{N}{24\pi} \int \sqrt{g}\, R = -\frac{N}{6}\, \chi(\mathcal{M}). \tag{28}$$

7 Conclusion

We introduced sigma models whose base is a statistical manifold endowed with the Fisher-Rao metric, so that geometry is fixed by statistical distinguishability rather than imposed ad hoc. For five prototypical families we computed the Fisher metrics, action densities, and equations of motion explicitly, finding constant curvatures spanning flat, positive, and negative cases (Table 1). In particular, the Gaussian base realizes $\mathbb{H}^2$ ($R = -1$), the N=3 categorical simplex is positively curved ($R = +1/2$), while the shifted exponential and Gumbel bases furnish further negatively curved examples.

On the quantum side, we highlighted that N free target scalars on a fixed two-dimensional Fisher base exhibit a Weyl anomaly with density set by the base curvature, Equation (23), and an integrated anomaly controlled topologically by Gauss Bonnet. A covariant background-field expansion in normal coordinates (Section 4.2) organizes counterterms into curvature tensors; for constant-curvature bases the absence of ∇R terms makes perturbation theory particularly simple. We clarified that, unlike string theory, no "critical N" arises here because the base metric is not dynamical.

Structurally, exponential-family models provide a clean bridge between statistics and dynamics: constant kinetic coefficients occur if the log-partition is quadratic (Proposition 1), implying a locally flat base and vanishing local anomaly density. Physically, this appears as a constant inertia after field/coordinate rescalings, whereas generically the Fisher weight produces position-dependent inertia (clear in the Bernoulli line) and genuine anisotropy in $d \geq 2$ where curvature obstructs flattening.

Outlook. Several directions follow naturally: (i) higher-loop structure and curvature-squared terms on Fisher bases; (ii) supersymmetric and fermionic extensions; (iii) gauging base isometries and coupling to target-space gauge fields; (iv) data-driven applications where measured distributions directly fix the base geometry and hence the field dynamics. Together, these suggest a program where emergent phenomena are modeled from information-theoretic inputs, with constant-curvature Fisher manifolds serving as tractable testbeds.

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